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Football as a Differential Game

J. V. Breakwell*
Stanford University, Stanford, California 94035
and
A. W. Merz†
Lockheed Missiles and Space Company, Palo Alto, California 94304

Introduction

AMERICAN football provides an analogy to the aerial defense problem. The performance criterion in football is the distance upfield moved by a specific offensive player carrying the football, while the aerial defense problem instead involves a number of offensive players, and a "field" that is only roughly rectangular. A more accurate analogy might replace the goal line with a number of "goal points." But in both problems, the evader seeks to maximize the downrange distance covered before being tackled, or "intercepted," by a pursuer. In football, the evader's teammates are of interest to the pursuers because one of them may become the evader if the ball is passed to him by the initial evader. This feature is also evidently absent from the equivalent aerial defense problem.

The defensive pursuer team wants to minimize this upfield yardage by tackling the evader with the football. Tackling is modeled here as a range constraint; if the range from this evader to any pursuer falls below the *capture* range, the evader has been tackled and the play ends. But when a pass by the ball carrier is feasible, the pursuers must consider all evaders as potential receivers.

The control of any player is the direction in which he runs. Each team has a unique optimal tactic for most geometric configurations. Midfield (mirror) symmetry obviously permits a right-left choice of tactics for the evader. Multiple tactics occur more generally at *dispersal* points,[1] but in nonsymmetric configurations they are less apparent. The evader speeds are treated here as equal to or greater than the pursuer speeds.

In this game, tactics depend on both relative and real geometry of the players. This effectively doubles the order of the problem. Every player on the two teams has coordinates (x, y) so American football is a system of order $2 \times 2 \times 11 = 44$, if speeds are constant and turns are immediate. A complete solution for the game is not given here (!), but three idealized subcases are solved. These are as follows:

1) The one-on-one equal-speed problem. The question answered is: How should evader E run to maximize the distance upfield, and how should defending pursuer P run to minimize this distance?

2) The three-on-three equal-speed problem. Tactics of the players are required as functions of six pairs of (x,y) coordinates. This version permits the ball to be passed from E_1 to either E_2 or E_3, and the pursuers must defend against both possibilities.

3) The one-on-one problem when E is faster than P. Now E has the prospect of running "around" P, and curved paths result. When P is faster, optimal paths are straight.

One-on-One Tactics, Equal Speeds

The geometry at any time is shown in Fig. 1. For any positions (x_p,y_p) and (x_e,y_e), there exists a hyperbolic locus passing between the players such that the time needed for E to arrive at any point on the locus equals the time required by P to arrive within the tackle range L of the same point. The point toward which they should run is that which is farthest upfield. This point is (x_f,y_f), and whether x_f is zero or positive depends on x_e, $\Delta x = x_p - x_e$, $\Delta y = y_p - y_e$, and the tackle range L.

The two cases are shown in Fig. 2. A midfield tackle is optimal if

$$\Delta y > 0$$

$$\Delta x < L$$

$$\delta x < x_e \qquad (1)$$

where δx is the x distance from E to the apex of the hyperbola,

$$\delta x = (\Delta x/2)(\Delta y/D - 1)$$

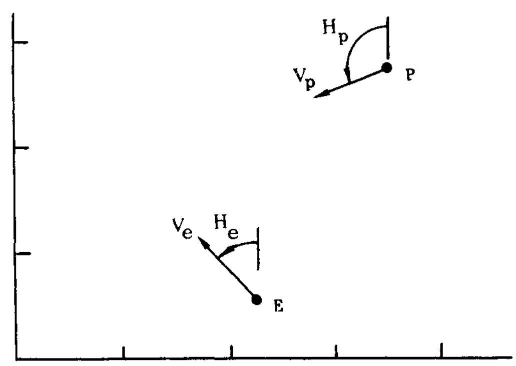

Fig. 1 General one-on-one kinematics.

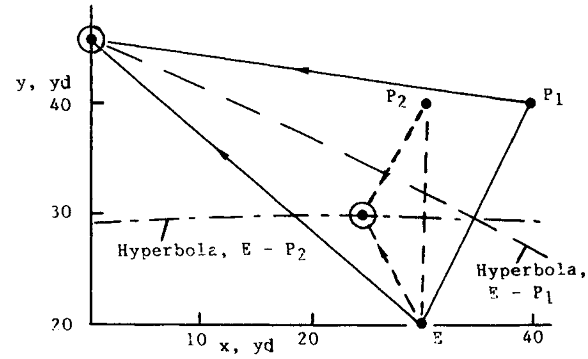

Fig. 2 Hyperbolic loci in the equal-speed case.

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*Professor, Aeronautics and Astronautics. Fellow AIAA (deceased).

†Staff Scientist, Advanced Systems Studies. Associate Fellow AIAA.

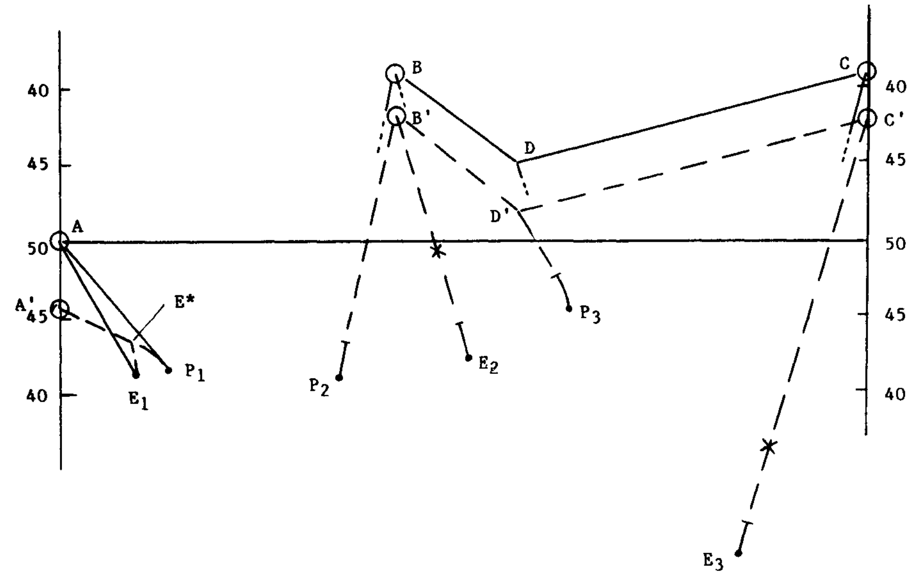

Fig. 3 Optimal and suboptimal three-on-three play.

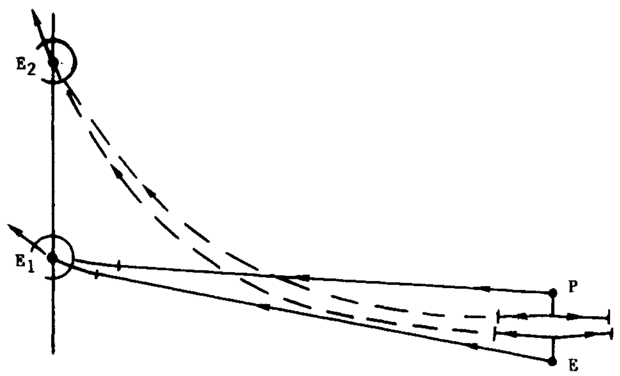

Fig. 4 Optimal one-on-one play, faster evader.

and where

$$D = \sqrt{L^2 - \Delta x^2}$$

When $\delta x < x_e$, E is closer to the maximum of the hyperbola than to the field boundary. In this case, the heading angles are symmetric with respect to the cross-field direction. These optimal heading angles for P and E are $H_p = \pi - H_1$, and $H_e = H_1$, where $\sin H_1 = \Delta x/L$. These headings provide a "minimax" of the distance upfield for P and E for the short-range, nearly "head-on," initial geometry.

In the simpler boundary end condition, one or more of the constraints of Eq. (1) are not met, and the players head toward the point $(0,y_f)$, where y_f is the solution to a quadratic equation. Two example trajectories are shown in Fig. 2 for $R = 1$ yd. The hyperbolic loci are nearly linear because R is much less than the initial range. The slope, which is greater at the sideline, indicates the sensitivity to wrong headings.

Three-on-Three Tactics, Equal Speeds

The six players shown in Fig. 3 are a generalization of the case just discussed. Here, E_1 has the ball, and the other two evaders must be optimally defended against by the other two pursuers. The endpoint for E_1 is the point A, at the left boundary. At this time, E_1 passes the ball to either of his teammates.

Both P_2 and P_3 influence E_2's motion while P_3 must cover both E_2 and E_3. The point A is defined by the condition $|P_1 - A| = |E_1 - A| + L$. P_3 runs to point D, while E_1 runs to A, at which time P_3 runs toward either B or C, depending on the pass receiver chosen. Points B, C, and D are determined by the length conditions,

$$|P_3 - D| = |E_1 - A|$$

$$|P_3 - D| + |D - B| = |E_2 - B| + L = |P_2 - B|$$

$$|P_3 - D| + |D - C| = |E_3 - C| + L \tag{2}$$

and by

$$y(B) = y(C) \tag{3}$$

If nonoptimal tactics are used by one player, curved paths may be taken by the others. This is also shown in Fig. 3, where E_1 is shown as following a more nearly upfield heading until the range is reduced to L. He then moves directly away from P_1 until the boundary is reached. The other paths are then straight after this contact by P_1. The time of receipt of the ball by E_2 or E_3 is shown by "x" on their trajectories, and the loss in yardage due to incorrect play by E_1 is about 3 yd.

One-on-One, Faster Evader

If E is faster than P, then $w = V_e/V_p > 1$, and a "safe-contact" interval can result, such that the range exactly equals the tackle range for a finite time. As in Isaacs' "deadline game,"[1] the faster E can then run around P, and both motions are curved; the paths are, respectively, epicycloidal and hypocycloidal.[2] The evader can pass by the pursuer if the ratio of field width to tackle radius satisfies the relation

$$W/L > 2w[\sqrt{w^2 - 1} + \arcsin(1/w) + \pi/2]/(w^2 - 1) \tag{4}$$

Football fields are 53 yd wide, so $W/L = 53$ implies that E can pass P if the speed ratio exceeds $w = 1.061$. For any smaller speed advantage, E is tackled at the boundary. Figure 4 shows the situation if $w = 1.059$, and the players are initially 3.8 yd apart at midfield, with P directly ahead of E. If E is obliged (by other players, etc.) to break to the left, E and P follow optimal straight paths that blend smoothly into their cycloidal paths; E ends at E_1. The composite paths are rotated from Isaacs' description by the angle which E's path makes with the boundary at E_1. This angle is determined by the initial geometry and is here about 60 deg.

However, when E can break to either side, he can do better by running directly forward for a time, as in the "Cornered Rat" game, before breaking to one side or the other. This short duration is such that a right switch following closely on an initial left break by E ceases to be effective. The pursuer of course initially runs directly toward the evader, and in this case E breaks to one side or the other when the separation is only slightly more than L. The subsequent composite motions end with E at E_2, where his heading makes only about 10 deg with the sideline. E has gained about 10 yd compared with the final position E_1. The diagram here indicates that the players are in contact at the tackle range over the majority of the time, as shown by the curved dashed lines.

Conclusions

Football is a multiplayer game analogous to the continental air defense problem. Real-space guidance constraints in such high-order games can be accounted for if the dynamics are simple, and differential-game mini-max optimization is feasible. Mathematical complexities occur if the evaders are faster than the pursuers.

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Asymptotic Disturbance Rejection for Momentum Bias Spacecraft

Christopher D. Rahn*
Space Systems/Loral, Palo Alto, California 94303

Introduction

During the normal operation of momentum bias spacecraft, a control system maintains the pitch axis perpendicular to the orbit plane, the roll axis pointing along the orbital velocity vector, and the yaw axis pointing at the Earth's center. At least one momentum wheel spins to provide an angular momentum bias along the pitch axis.[1-3] This pitch momentum is varied to control the pitch Euler angle of the spacecraft. Varying the wheel momentum along the yaw axis controls the roll and yaw Euler angles. An Earth sensor measures the roll and pitch angles. Yaw, though not directly measured, is observable from roll.

The accuracy to which the spacecraft maintains its prescribed attitude depends on the size and nature of the environmental disturbance torques. A previous technique[4] improved pointing performance by estimating one component of these disturbance torques. This Note introduces a method of achieving asymptotic disturbance rejection through disturbance torque estimation and feedback. The method uses the Earth sensor roll signal, a full-order observer, a disturbance torque estimator, and magnetic torquers.

Roll/Yaw Dynamics

Dougherty et al.[2] determined the linearized equations of motion for a momentum bias spacecraft. They showed that the pitch dynamics are decoupled from the roll/yaw dynamics. Terasaki[1] derived the equations of motion for a system with yaw momentum storage. Lebsock[4] assumed that the relatively high-frequency roll/yaw nutation dynamics are damped by wheel control[1,3-5] and neglected gravity gradient torques to generate the simplified momentum dynamics:

$$\dot{x} = Ax + Bu \tag{1}$$

with

$$a = \begin{bmatrix} H_x \\ H_z \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & \omega_o \\ -\omega_o & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad u = \begin{bmatrix} M_x \\ M_z \end{bmatrix} \tag{2}$$

where H_x and H_z are the body axes roll and yaw angular momentum components, respectively; $\omega_o > 0$ is the orbit rate; and M_x and M_z are the body axes torques. The body axes torques equal the environmental disturbance torques T_{dx} and T_{dz} minus the magnetic control torques u_x and u_z,

$$M_x = T_{dx} - u_x, \qquad M_z = T_{dz} - u_z \tag{3}$$

The attitude errors, roll ϕ and yaw ψ, are related to the angular momentum of the spacecraft by

$$\phi = \frac{h_z - H_z}{H_n}, \qquad \psi = \frac{H_x}{H_n} \tag{4}$$

where h_z is the yaw momentum stored in the wheels and $H_n > 0$ is the momentum bias. Note that the yaw error is proportional to the roll momentum and the roll error is proportional to the net yaw momentum.

For geosynchronous spacecraft, solar torques are the dominant environmental torques. The rotation of the spacecraft bus with respect to the sun causes the solar torques to vary with orbital position. The solar torques acting on a particular spacecraft are estimated by numerical integration of solar pressure over a surface model of the spacecraft at several orbital positions. Then the coefficients of a truncated Fourier series for the resultant roll and yaw torques are calculated, giving

$$T_{dx}(t) = \frac{1}{2}A_{x0} + \sum_{n=1}^{n_{\max}} [A_{xn} \cos(n\omega_o t) + B_{xn} \sin(n\omega_o t)$$

$$T_{dz}(t) = \frac{1}{2}A_{z0} + \sum_{n=1}^{n_{\max}} [A_{zn} \cos(n\omega_o t) + B_{zn} \sin(n\omega_o t)] \tag{5}$$

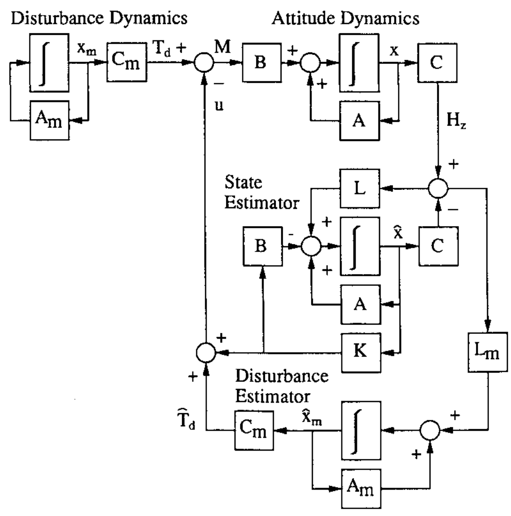

Fig. 1 Control system block diagram.

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*Research and Development Engineer, 3825 Fabian Way.